Hilroy

80 Pages
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EXERCISE BOOK
CAHIER D'EXERCICES

NAME/NOM ____________

SUBJECT/SUJET MATH 256

12107

*get textbook
* youtube material/LEC => maybe

Ordinary Differential Equations (ODE's)   9/7/18

1st order Linear ODE

$a(t)y' + b(t)y = c(t)$ => GENERAL Form

$y' + \frac{b(t)}{a(t)}y = \frac{c(t)}{a(t)}$ => STANDARD Form

ex. 1

$mV' = mg - \lambda V$ | $y(t) = v(t)$ $a=1$ $b = \frac{\lambda}{m}$ $g = c(t)$

$v' + \frac{\lambda}{m}v = g$

$y' + \frac{b(t)}{a(t)}y = \frac{c(t)}{a(t)}$ *note $\frac{b(t)}{a(t)} = p(t)$

$y' + p(t)y = g(t)$ => STANDARD

if $p(t) = 0$
$y' = g(t) \Rightarrow y = \int g(t)\,dt$

if $p(t) \neq 0$ //Multiply an integrating factor

$\mu(t) = e^{\int p(t)\,dt}$

so.....

$\mu(t)y' + p\mu(t)y = \mu g$

$\boxed{\text{Example}}$

$$y' - \frac{y}{2t} = \frac{y^2}{2t} \; ; \; y(1) = 1/2$$

Soln $\Rightarrow$ Bernoullis with $n = 2$

let $v = y^{1-n}$
$= y^{1-2}$
$= y^{-1}$

$$v' + \frac{1}{2t} v = -\frac{1}{2t}$$

$$N = e^{\int \frac{1}{2t} dt} = \sqrt{t}$$

$$\int Ng = \int \sqrt{t}\left(\frac{-1}{2t}\right) dt = -\sqrt{t}$$

$$v = \frac{1}{N}\left(C + \int Ng\right) = \frac{1}{\sqrt{t}}\left(C - \sqrt{t}\right)$$

$y(1) = \frac{1}{2}$ $\qquad$ $v(1) = \frac{1}{y(1)} = 2$

$\therefore C = 3$

$\boxed{\text{Interval of Existence}}$ By IC

$$v = \frac{1}{\sqrt{t}}(3 - \sqrt{t})$$

$$y = \frac{1}{v} = \frac{\sqrt{t}}{3 - \sqrt{t}}$$

$t > 0$, $3 - \sqrt{t} \neq 0$ $\therefore t \neq 9$

So.....

$0 < t < 9$

$\boxed{\text{Example}}$ $\frac{dy}{dx} = \frac{y^2}{x^2+y^2}$

$x \to \lambda x$ $\quad y = xv$

$y \to \lambda y$ $\quad y' = xv' + v$

$$= \frac{x^2 v^2}{x^2 + x^2 v^2}$$

$$= \frac{v^2}{1+v^2}$$

$$xv' = \frac{v^2}{1+v^2} - v = \frac{v^2 - v - v^3}{1+v^2} \leftarrow \text{SEPERABLE}$$

$$\frac{(1+v^2)\,dv}{(v^2-v-v^3)} = \frac{1}{x}\,dx$$

$\boxed{\text{Possible ODE's}}$

1. $\frac{dy}{dt} + p(t)y = g(t)$
2. $\frac{dy}{dt} + p(t)y = g(t)\,y^n$ // $(v = y^{1-n})$
3. $\frac{dy}{dx} = f(x)\,g(y)$
4. $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ // $v = \frac{y}{x}$

Summary

Factors that determine interval of existence

1. initial condition t_0
2. Equation
3. Solution

Example

$$y' = \frac{t^2}{y^2-1}; \quad y(0)=0$$

Sol'n

$$(y^2-1)\,dy = t^2\,dt$$

$$\frac{y^3}{3} - y = \frac{t^3}{3} + C$$

$$0 - 0 = 0 + C$$

$$C = 0$$

$$y^3 - 3y = t^3$$

1. $t_0 = 0$
2. $y \neq \pm 1$
3. if $y = 1 \Rightarrow t^3 = 1 - 3$

$$t^3 = -2$$

$$t \neq \sqrt[3]{-2}$$

if $y = -1 \Rightarrow t^3 = -1 + 3$

$$t^3 = 2$$

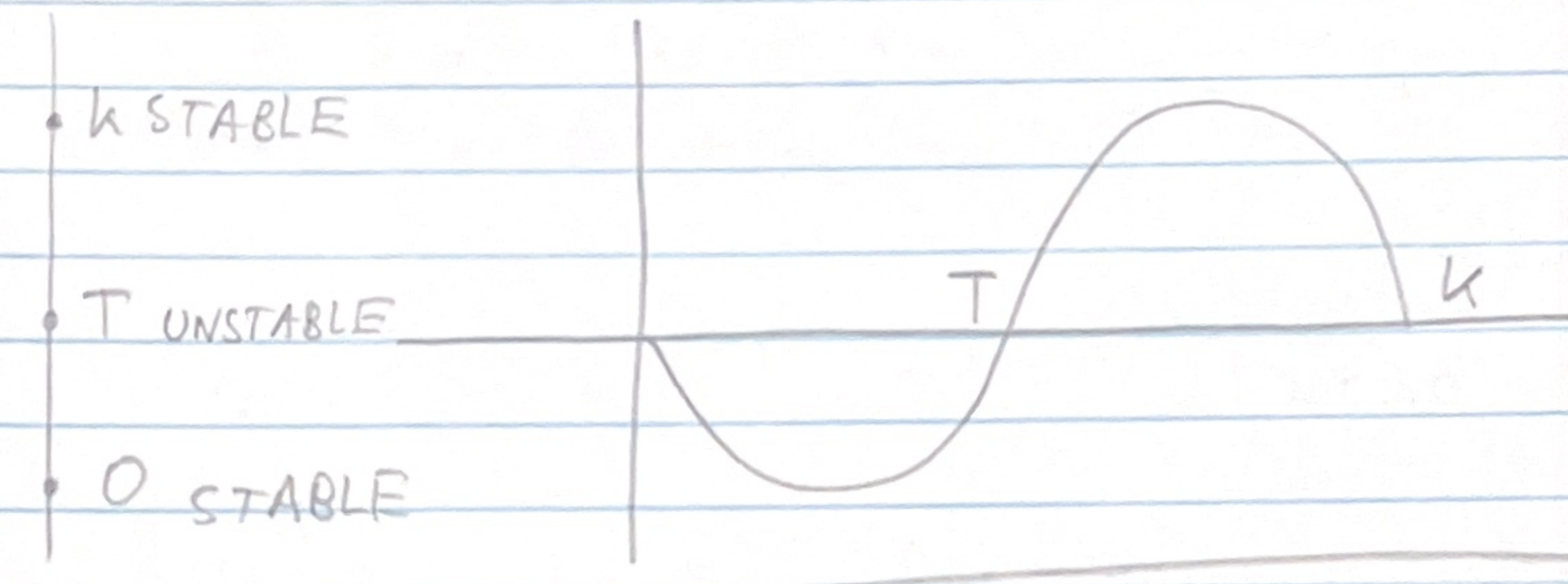
$$t \neq \sqrt[3]{2}$$

Interval of Existence

$$\sqrt[3]{-2} < t < \sqrt[3]{2}$$

$\boxed{\text{Model 4}}$ LOGISTIC WITH THRESHOLD

$$\frac{dy}{dt} = -ry\left(1-\frac{y}{T}\right)\left(1-\frac{y}{k}\right) \quad 0<T<k$$



Chapter 3 2^{nd} Order ODE

$y'' = f(t, y, y')$ GENERAL FORM

$y(t_0) = y_0$

$y_1'(t_0) = y_1$

$p(t)y'' + Q(t)y' + R(t)y = G(t)$ GENERAL

*use this $\longrightarrow$ $y'' + p(t)y' + q(t)y = g(t)$ STANDARD

$\boxed{\text{Case 4}}$ $r_1 = r_2$

if r is a eigenvalue then be^{rt} may not be a sol'n unless $b = \vec{a}$

eigenvector

$$x^{(1)} = a^{(1)} e^{rt}$$

$$x^{(2)} = ta^{(1)} e^{rt} + \underline{be^{rt}}$$

may not be a sol'n if $b \neq a^{(1)}$

$\boxed{\text{Example}}$

$$x' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} x$$

$$\det(P - rI) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (r-1)(r-3)+1$$

$$r^2 - 4r + 4 = (r-2)^2 = 0$$

$$r_1 = r_2 = 2$$

For $r_1 = r_2 = 2$

$$\begin{pmatrix} 1-2 & -1 \\ 1 & 3-2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-a_1 - a_2 = 0$$
$$a_1 + a_2 = 0$$

choose $a_1 = 1$ $\quad a^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$x^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$

Solving eqn 1 :

$$\vec{a} = k\begin{pmatrix}1\\-2\end{pmatrix}, k \text{ is a constant}$$

Plug into 2 :

$$k\begin{pmatrix}1\\-2\end{pmatrix} - \begin{pmatrix}b_1\\b_2\end{pmatrix} = \begin{pmatrix}1 & 1\\4 & 1\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$$

$$k - b_1 = b_1 + b_2 \longrightarrow 2b_1 + b_2 = k$$
$$-2k - b_2 = 4b_1 + b_2 + 1 \longrightarrow 4b_1 + 2b_2 = -(2k+1)$$

$$\left.\begin{array}{l}\begin{cases}2b_1 + b_2 = k\\4b_1 + 2b_2 = -2k-1\end{cases}\\ \\ \begin{cases}2b_1 + b_2 = k\\2b_1 + b_2 = -k - \frac{1}{2}\end{cases}\end{array}\right\} \text{equal}$$

$$k = -k - \tfrac{1}{2} \longrightarrow k = 1/4$$

$$2b_1 + b_2 = -1/4 \quad \text{choose } b_1 = 0$$

$$\boxed{b_2 = -1/4}$$

$$X_p = t\,(-1/4)\begin{pmatrix}1\\-2\end{pmatrix}e^{-t} + \begin{pmatrix}0\\-1/4\end{pmatrix}e^{-t}$$

$$\frac{s+5}{s^2+3s+2} = \frac{s+5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$\frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$$

$$A(s+2)+B(s+1) = s+5 \rightarrow A+B=1$$

$$2A+B=5$$

$$\therefore A=4$$

$$B=-3$$

$$e^{rt} \rightarrow \frac{1}{s-r}$$

$$e^{-t} \rightarrow \frac{1}{s+1}$$

$$\frac{s+5}{s^2+3s+2} = \frac{4}{s+1} - \frac{3}{s+2}$$

$$\downarrow \qquad \downarrow$$

$$4e^{-t} \quad -3e^{-2t}$$

$$\frac{1}{(s-2)(s^2+3s+2)} = \frac{1}{(s-2)(s+1)(s+2)} = \frac{C}{s-2} + \frac{D}{s+1} + \frac{E}{s+2}$$

$$\frac{C(s+1)(s+2) + D(s-2)(s+2) + E(s-2)(s+1)}{(s-2)(s+1)(s+2)} = \frac{1}{(s-2)(s+1)(s+2)}$$

$$C+D+E=0$$

$$3C-E=0$$

$$2C-4D-2E=1$$

$$\therefore C = 1/12$$

$$D = -1/3$$

$$E = 1/4$$

Laplace Transformations

10/29/18
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Example

$y'' + y = \sin(2t)$; $y(0) = 2$, $y'(0) = 1$

sol'n:

$$Y(s) = L[y](s)$$

$$s^2 Y(s) - sy(0) + Y(s) = L[\sin(2t)]$$

Partial Fractions

$$\frac{1}{(s-\lambda)^2 + N^2} \rightarrow \frac{A(s-\lambda)}{(s-\lambda)^2 + N^2} + \frac{B}{(s-\lambda)^2 + N^2}$$

Example

WebWork PF

$$\frac{s^6 + 5s + 8}{(s-2)^2 (s+1)(s^2+4)((s-1)^2+1)} = Y(s)$$

$$\frac{A}{(s-2)^2} + \frac{B}{(s-2)} + \frac{C}{(s+1)} + \frac{Ds}{(s^2+4)} + \frac{E}{(s^2+4)} + \frac{F(s-1)}{(s-1)^2+1} + \frac{G}{(s-1)^2+1}$$

$$Y(t) = Ate^{2t} + Be^{2t} + Ce^{-t} + D\cos(2t) + \frac{1}{2}E\sin(2t) + Fe^{t}\cos(t) + Ge^{t}\sin(t)$$

$$g(t) = 2H(t) \quad 0 \le t < 4$$
$$+3H(t-4) \quad 4 \le t < 7$$
$$-6H(t-7) \quad 7 \le t < 9$$
$$+2H(t-9) \quad t \ge 9$$

$$L[g] = \frac{2}{s} + \frac{3}{s}e^{-4s} + \frac{6}{s}e^{-7s} + \frac{2}{s}e^{-9s}$$

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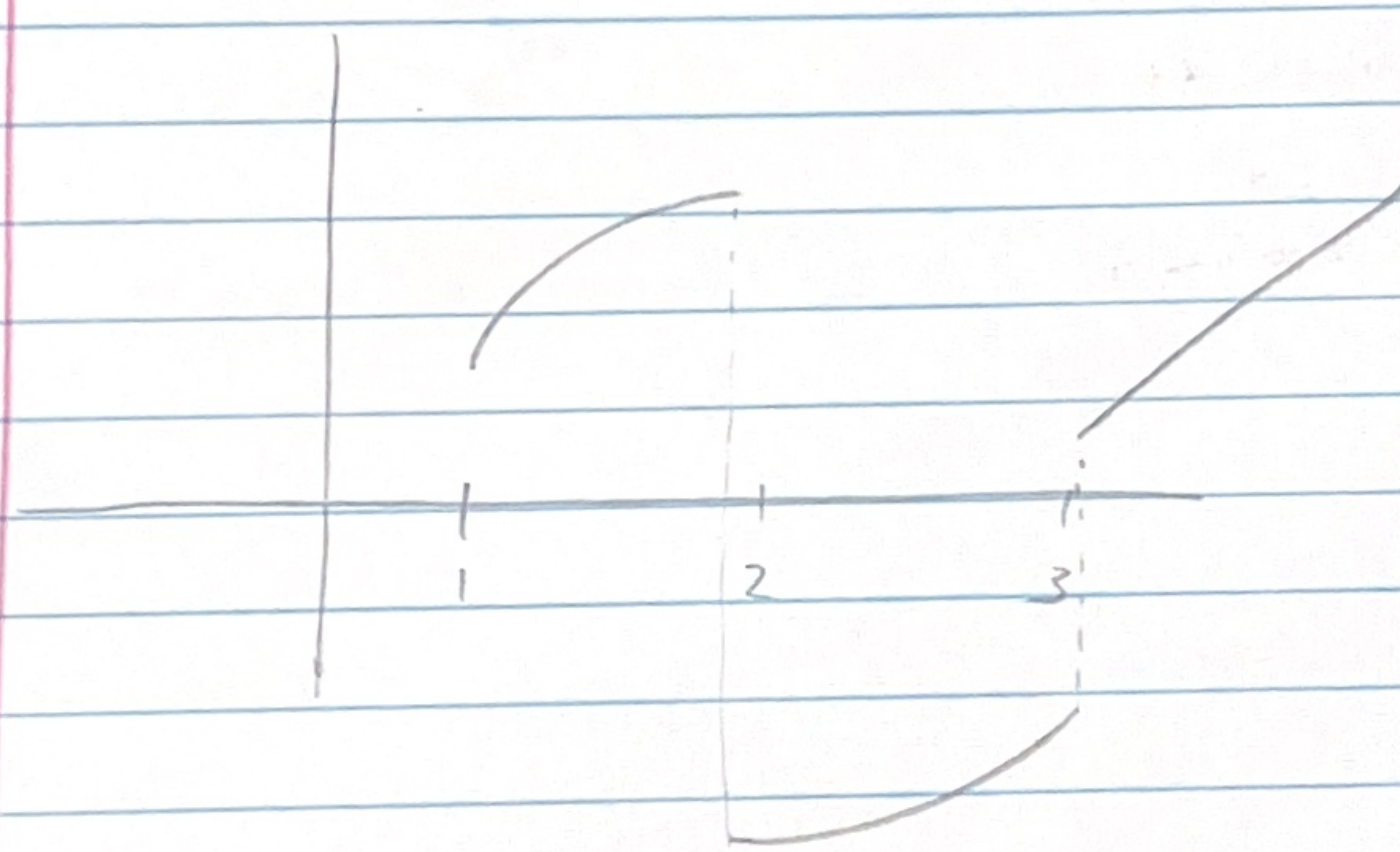
Heaviside Function

Piecewise Continuous



- Piecewise, no longer constant but still continuous, "On intervals"

Formula

$$L[H(t-c)f(t-c)]$$

$$= e^{-cs} L[f](s)$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 3[s\,Y(s) - y(0)] + 2Y(s) = \mathcal{L}[g]$$

$$(s^2+3s+2)Y(s) = e^{-s}\cdot\frac{1}{s} - e^{-2s}\cdot\frac{1}{s}$$

$$Y(s) = \frac{1}{(s)(s+1)(s+2)} = (e^{-s} - e^{-2s})$$

Partial Fractions

$\downarrow$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\frac{A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1)}{(s)(s+1)(s+2)}$$

$A+B+C=0$

$3A+2B+C=0$

$2A+0B+0C=1$

$\therefore A = \frac{1}{2}$

$B = -1$

$C = \frac{1}{2}$

$$Y(s) = \left(\frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}\right) = \left(e^{-s} - 2e^{-2s}\right)$$

$$\frac{1}{2}e^{-s}\cdot\frac{1}{s} - \frac{1}{2}e^{-2s}\cdot\frac{1}{s} - \frac{1}{s+1}e^{-s} + \frac{1}{s+1}e^{-2s} + \frac{1}{2}e^{-s}\cdot\frac{1}{s+2} + \frac{1}{2}e^{-2s}\frac{1}{s+2}$$

Example

$2y'' + y' + 2y = \delta(t-5)$; $y(0) = 0$, $y'(0) = 0$

Sol'n :

$(s^2 Y + s + 2) Y = e^{-5s}$

Note:

$Y = \dfrac{e^{-5s}}{2s^2 + s + 2}$

$e^{\lambda t} \sin(\omega t) = \dfrac{\omega}{(s-\lambda)^2 + \omega^2}$

$Y = \dfrac{1}{2} e^{-5s} \cdot \dfrac{1}{s^2 + \frac{1}{2}s + 1}$

$\boxed{\text{Convolutions}}$

$$f * g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

Example

$$\cos(t) * 1 = \int_0^t \cos(t-\tau)\, 1\, d\tau$$

$$= -\sin(t-\tau) \Big|_{\tau=0}^{\tau=t} = \sin(t)$$

11/08/16
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Two-Point Boundary Value Problems

Eigenvalue problem

$$\begin{cases} y'' + \lambda y = 0, & 0 < t < \pi \\ y(0) = 0, & y(\pi) = 0 \end{cases}$$

Case 1 $\lambda < 0$

Case 2 $\lambda = 0$

Case 3 $\lambda > 0$, $\lambda = n^2$ $y = C\sin(nt)$
where $n = 1, 2, 3, \ldots$

Example $y'' + \lambda = 0$

rewrite $\lambda = -a^2$ for $a > 0$

$r^2 - a^2 = 0$

$\rightarrow y = C_1 e^{-at} + C_2 e^{at}$

General

$$\begin{cases} y'' + \lambda y = 0 \; ; \; 0 < t < L \\ y(0) = y(L) = 0 \end{cases}$$

$\lambda = \left(\frac{n\pi}{L}\right)^2$ $y = \sin\left(\frac{n\pi}{L} t\right)$

Room → SCRF 100

Periodic Boundary Conditions

$$\begin{cases} y'' + \lambda y = 0; \quad -L < x < L \\ y(-L) = y(L) \\ y'(-L) = y'(L) \end{cases}$$

① Dirchlet

② Neumann

③ Periodic Bandary condition

Midterm :

Chapter 4.1, 6, 7

Fourier Series Expansion

Periodic Bandary Conditions : 2L-periodic eigenfunctions

Eigenfunctions: $1, \cos\frac{\pi}{L}x, \sin\frac{\pi}{L}x, \cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x$

$\vdots$

$\cos\frac{n\pi}{L}x, \sin\frac{n\pi}{L}x$

$\boxed{Q}$ given a 2L-periodic function $f(x+2L)=f(x)$

expand $f(x)$ in terms of

$$\left\{1, \cos\frac{n\pi}{L}x, \sin\frac{n\pi}{L}x\right\}_{n=1}^{\infty}$$

③ Sum up

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,0) = \phi(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where $b_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)dx$ — Solution

Inhomogenous Boundary Conditions. "Cant use separation of Variables"

$$\begin{cases} u_t = k u_{xx}, & 0 < x < L, \ t > 0 \\ u(0,t) = T_0 & u(L,t) = T_1 \\ u(x,0) = \phi(x) \end{cases}$$

T_1, T_0 are constants

How to solve?

Sol'n:

"Method of shifting Data"

Heat Equation

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For Heat eqn
Dirichlet → Fourier Sine
Neuman → Fourier Cosine
Periodic → Full Fourier

Heat eqn with Neumann BVC

$$\begin{cases} u_t = k u_{xx} \\ u_x(0,t) = 0, \quad u_x(L,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Step 1: method of separation of variable

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(L) = 0 \end{cases} \qquad T' + k\lambda T = 0$$

Step 2:

$$X'' + \lambda X = 0, \quad X'(0) = X'(L) = 0$$

$$\lambda_0 = 0 \qquad X_0 = 1 \longrightarrow T_0 = \text{constant}$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad X_n = \cos\left(\frac{n\pi x}{L}\right) \rightarrow T_n = Ce^{-k\lambda_n t}$$

Fourier Cosine Series

Step 3: Sum Up

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Asymptotic Behavior $t \to \infty$ $\quad u(x,t) \to \dfrac{a_0}{2} = \dfrac{1}{L}\displaystyle\int_0^L \phi(x)\,dx$